Chapter 13

# LIMITS AND DERIVATIVES

## 13.1 Overview

### 13.1.1 *Limits of a function*

Let $f$ be a function defined in a domain which we take to be an interval, say, I. We shall study the concept of limit of $f$ at a point '$a$' in I.

We say $\lim\limits_{x \to a^-} f(x)$ is the expected value of $f$ at $x = a$ given the values of $f$ near to the left of $a$. This value is called the *left hand limit* of $f$ at $a$.

We say $\lim\limits_{x \to a^+} f(x)$ is the expected value of $f$ at $x = a$ given the values of $f$ near to the right of $a$. This value is called the *right hand limit* of $f$ at $a$.

If the right and left hand limits coincide, we call the common value as the limit of $f$ at $x = a$ and denote it by $\lim\limits_{x \to a} f(x)$.

**Some properties of limits**

Let $f$ and $g$ be two functions such that both $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist. Then

(i) $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

(ii) $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

(iii) For every real number $\alpha$

$\lim\limits_{x \to a} (\alpha f)(x) = \alpha \lim\limits_{x \to a} f(x)$

(iv) $\lim\limits_{x \to a} [f(x)\, g(x)] = [\lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x)]$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}, \text{ provided } g(x) \neq 0$$

**Limits of polynomials and rational functions**

If $f$ is a polynomial function, then $\lim\limits_{x \to a} f(x)$ exists and is given by

$$\lim_{x \to a} f(x) = f(a)$$

**An Important limit**

An important limit which is very useful and used in the sequel is given below:

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

***Remark*** The above expression remains valid for any rational number provided '$a$' is positive.

**Limits of trigonometric functions**

To evaluate the limits of trigonometric functions, we shall make use of the following limits which are given below:

$$\text{(i)} \quad \lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \text{(ii)} \quad \lim_{x \to 0} \cos x = 1 \qquad \text{(iii)} \quad \lim_{x \to 0} \sin x = 0$$

**13.1.2** ***Derivatives*** Suppose $f$ is a real valued function, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{... (1)}$$

is called the **derivative** of $f$ at $x$, provided the limit on the R.H.S. of (1) exists.

**Algebra of derivative of functions** Since the very definition of derivatives involve limits in a rather direct fashion, we expect the rules of derivatives to follow closely that of limits as given below:

Let $f$ and $g$ be two functions such that their derivatives are defined in a common domain. Then :

(i) Derivative of the sum of two function is the sum of the derivatives of the functions.

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

(ii) Derivative of the difference of two functions is the difference of the derivatives of the functions.

$$\frac{d}{dx}\left[f(x) - g(x)\right] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

(iii) Derivative of the product of two functions is given by the following *product rule*.

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = \left(\frac{d}{dx}f(x)\right)\cdot g(x) + f(x)\cdot\left(\frac{d}{dx}g(x)\right)$$

This is referred to as Leibnitz Rule for the product of two functions.

(iv) Derivative of quotient of two functions is given by the following *quotient rule* (wherever the denominator is non-zero).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left(\frac{d}{dx}f(x)\right)\cdot g(x) - f(x)\cdot\left(\frac{d}{dx}g(x)\right)}{\left(g(x)\right)^2}$$

## 13.2 Solved Examples

### Short Answer Type

**Example 1** Evaluate $\lim\limits_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right]$

**Solution** We have

$$\lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right] = \lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x(x-1)(x-2}\right]$$

$$= \lim_{x\to 2}\left[\frac{x(x-1)-2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{x^2-5x+6}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right] \quad [x-2\neq 0]$$

$$= \lim_{x\to 2}\left[\frac{x-3}{x(x-1)}\right] = \frac{-1}{2}$$

**Example 2** Evaluate $\lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$

**Solution** Put $y = 2 + x$ so that when $x \to 0$, $y \to 2$. Then

$$\lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} = \lim_{y \to 2} \frac{y^{\frac{1}{2}} - 2^{\frac{1}{2}}}{y - 2}$$

$$= \frac{1}{2}(2)^{\frac{1}{2} - 1} = \frac{1}{2} \cdot 2^{-\frac{1}{2}} = \frac{1}{2\sqrt{2}}$$

**Example 3** Find the positive integer $n$ so that $\lim_{x \to 3} \frac{x^n - 3^n}{x - 3} = 108$.

**Solution** We have

$$\lim_{x \to 3} \frac{x^n - 3^n}{x - 3} = n(3)^{n-1}$$

Therefore, $n(3)^{n-1} = 108 = 4\,(27) = 4(3)^{4-1}$

Comparing, we get $n = 4$

**Example 4** Evaluate $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

**Solution** Put $y = \frac{\pi}{2} - x$. Then $y \to 0$ as $x \to \frac{\pi}{2}$. Therefore

$$\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) = \lim_{y \to 0} [\sec(\frac{\pi}{2} - y) - \tan(\frac{\pi}{2} - y)]$$

$$= \lim_{y \to 0} (\operatorname{cosec} y - \cot y)$$

$$= \lim_{y \to 0} \left( \frac{1}{\sin y} - \frac{\cos y}{\sin y} \right)$$

$$= \lim_{y \to 0} \left( \frac{1 - \cos y}{\sin y} \right)$$

$$= \lim_{y \to 0} \frac{2\sin^2 \frac{y}{2}}{2\sin\frac{y}{2}\cos\frac{y}{2}} \qquad \left(\begin{array}{l}\text{since}, \sin^2\frac{y}{2} = \frac{1-\cos y}{2} \\ \sin y = 2\sin\frac{y}{2}\cos\frac{y}{2}\end{array}\right)$$

$$= \lim_{\frac{y}{2} \to 0} \tan\frac{y}{2} = 0$$

**Example 5** Evaluate $\lim_{x \to 0} \frac{\sin(2+x) - \sin(2-x)}{x}$

**Solution** (i) We have

$$\lim_{x \to 0} \frac{\sin(2+x) - \sin(2-x)}{x} = \lim_{x \to 0} \frac{2\cos\frac{(2+x+2-x)}{2}\sin\frac{(2+x-2+x)}{2}}{x}$$

$$= \lim_{x \to 0} \frac{2\cos 2\sin x}{x}$$

$$= 2\cos 2 \lim_{x \to 0} \frac{\sin x}{x} = 2\cos 2 \quad \left(\text{as} \lim_{x \to 0} \frac{\sin x}{x} = 1\right)$$

**Example 6** Find the derivative of $f(x) = ax + b$, where $a$ and $b$ are non-zero constants, by first principle.

**Solution** By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a(x+h) + b - (ax+b)}{h} = \lim_{h \to 0} \frac{bh}{h} = b$$

**Example 7** Find the derivative of $f(x) = ax^2 + bx + c$, where $a$, $b$ and $c$ are none-zero constant, by first principle.

**Solution** By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a(x+h)^2 + b(x+h) + c - ax^2 - bx - c}{h}$$

$$= \lim_{h \to 0} \frac{bh + ah^2 + 2axh}{h} = \lim_{h \to 0} ah + 2ax + b = b + 2ax$$

**Example 8** Find the derivative of $f(x) = x^3$, by first principle.

**Solution** By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + h^3 + 3xh(x+h) - x^3}{h}$$

$$= \lim_{h \to 0} (h^2 + 3x(x+h)) = 3x^2$$

**Example 9** Find the derivative of $f(x) = \frac{1}{x}$ by first principle.

**Solution** By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)$$

$$= \lim_{h \to 0} \frac{-h}{h(x+h)x} = \frac{-1}{x^2}.$$

**Example 10** Find the derivative of $f(x) = \sin x$, by first principle.

**Solution** By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}}{2 \cdot \frac{h}{2}}$$

$$= \lim_{h \to 0} \cos\frac{(2x+h)}{2} \cdot \lim_{h \to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= \cos x.1 = \cos x$$

**Example 11** Find the derivative of $f(x) = x^n$, where $n$ is positive integer, by first principle.

**Solution** By definition,

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{(x+h)^n - x^n}{h}$$

Using Binomial theorem, we have $(x + h)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}\, h + \dots + {}^nC_n\, h^n$

Thus, $$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{h\,(nx^{n-1} + \dots + h^{n-1}]}{h} = nx^{n-1}.$$

**Example 12** Find the derivative of $2x^4 + x$.

**Solution** Let $y = 2x^4 + x$

Differentiating both sides with respect to $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2x^4) + \frac{d}{dx}(x)$$

$$= 2 \times 4x^{4-1} + 1x^0$$

$$= 8x^3 + 1$$

Therefore, $$\frac{d}{dx}(2x^4 + x) = 8x^3 + 1.$$

**Example 13** Find the derivative of $x^2 \cos x$.

**Solution** Let $y = x^2 \cos x$

Differentiating both sides with respect to $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \cos x)$$

$$= x^2 \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2)$$

$$= x^2 (-\sin x) + \cos x\ (2x)$$

$$= 2x \cos x - x^2 \sin x$$

**Long Answer Type**

**Example 14** Evaluate $\lim\limits_{x \to \frac{\pi}{6}} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1}$

**Solution** Note that

$$2 \sin^2 x + \sin x - 1 = (2 \sin x - 1)(\sin x + 1)$$

$$2 \sin^2 x - 3 \sin x + 1 = (2 \sin x - 1)(\sin x - 1)$$

Therefore, $$\lim_{x \to \frac{\pi}{6}} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1} = \lim_{x \to \frac{\pi}{6}} \frac{(2\sin x - 1)(\sin x + 1)}{(2\sin x - 1)(\sin x - 1)}$$

$$= \lim_{x \to \frac{\pi}{6}} \frac{\sin x + 1}{\sin x - 1} \qquad \text{(as } 2 \sin x - 1 \neq 0\text{)}$$

$$= \frac{1 + \sin\frac{\pi}{6}}{\sin\frac{\pi}{6} - 1} = -3$$

**Example 15** Evaluate $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{\sin^3 x}$

**Solution** We have

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x} = \lim_{x\to 0}\frac{\sin x\left(\dfrac{1}{\cos x}-1\right)}{\sin^3 x}$$

$$= \lim_{x\to 0}\frac{1-\cos x}{\cos x\sin^2 x}$$

$$= \lim_{x\to 0}\frac{2\sin^2\dfrac{x}{2}}{\cos x\left(4\sin^2\dfrac{x}{2}\cdot\cos^2\dfrac{x}{2}\right)} = \frac{1}{2}.$$

**Example 16** Evaluate $\lim\limits_{x\to a}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

**Solution** We have $\lim\limits_{x\to a}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

$$= \lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\times\frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}}$$

$$= \lim_{x\to a}\frac{a+2x-3x}{\left(\sqrt{3a+x}-2\sqrt{x}\right)\left(\sqrt{a+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x\to a}\frac{(a-x)\left(\sqrt{3a+x}+2\sqrt{x}\right)}{\left(\sqrt{a+2x}+\sqrt{3x}\right)\left(\sqrt{3a+x}-2\sqrt{x}\right)\left(\sqrt{3a+x}+2\sqrt{x}\right)}$$

$$= \lim_{x\to a}\frac{(a-x)\left[\sqrt{3a+x}+2\sqrt{x}\right]}{\left(\sqrt{a+2x}+\sqrt{3x}\right)\left(3a+x-4x\right)}$$

$$= \frac{4\sqrt{a}}{3 \times 2\sqrt{3a}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}.$$

**Example 17** Evaluate $\lim_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - 1}$

**Solution** We have $\lim_{x \to 0} \frac{2\sin\left(\frac{(a+b)}{2}x\right) \sin \frac{(a-b)x}{2}}{2\frac{\sin^2 cx}{2}}$

$$= \lim_{x \to 0} \frac{2\sin \frac{(a+b)x}{2} \cdot \sin \frac{(a-b)x}{2}}{x^2} \cdot \frac{x^2}{\sin^2 \frac{cx}{2}}$$

$$= \lim_{x \to 0} \frac{\sin \frac{(a+b)x}{2}}{\frac{(a+b)x}{2} \cdot \left(\frac{2}{a+b}\right)} \cdot \frac{\sin \frac{(a-b)x}{2}}{\frac{(a-b)x}{2} \cdot \frac{2}{a-b}} \cdot \frac{\left(\frac{cx}{2}\right)^2 \times \frac{4}{c^2}}{\sin^2 \frac{cx}{2}}$$

$$= \left(\frac{a+b}{2} \times \frac{a-b}{2} \times \frac{4}{c^2}\right) = \frac{a^2 - b^2}{c^2}$$

**Example 18** Evaluate $\lim_{h \to 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

**Solution** We have $\lim_{h \to 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

$$= \lim_{h \to 0} \frac{(a^2 + h^2 + 2ah)[\sin a \cos h + \cos a \sin h] - a^2 \sin a}{h}$$

$$= \lim_{h \to 0} \left[\frac{a^2 \sin a (\cos h - 1)}{h} + \frac{a^2 \cos a \sin h}{h} + (h + 2a)(\sin a \cos h + \cos a \sin h)\right]$$

$$= \lim_{h\to 0}\left[\frac{a^2 \sin a\,(-2\sin^2 \frac{h}{2})}{\frac{h^2}{2}}\cdot\frac{h}{2}\right] + \lim_{h\to 0}\frac{a^2\cos a\sin h}{h} + \lim_{h\to 0}(h+2a)\sin(a+h)$$

$= a^2 \sin a \times 0 + a^2 \cos a\ (1) + 2a \sin a$

$= a^2 \cos a + 2a \sin a.$

**Example 19** Find the derivative of $f(x) = \tan(ax + b)$, by first principle.

**Solution** We have $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{\tan\big(a(x+h)+b\big)-\tan(ax+b)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{\sin(ax+ah+b)}{\cos(ax+ah+b)}-\frac{\sin(ax+b)}{\cos(ax+b)}}{h}$$

$$= \lim_{h\to 0}\frac{\sin(ax+ah+b)\cos(ax+b)-\sin(ax+b)\cos(ax+ah+b)}{h\cos(ax+b)\cos(ax+ah+b)}$$

$$= \lim_{h\to 0}\frac{a\sin(ah)}{a\cdot h\cos(ax+b)\cos(ax+ah+b)}$$

$$= \lim_{h\to 0}\frac{a}{\cos(ax+b)\cos(ax+ah+b)}\ \lim_{ah\to 0}\frac{\sin ah}{ah}\quad [\text{as } h\to 0\ \ ah\to 0]$$

$$= \frac{a}{\cos^2(ax+b)} = a\sec^2(ax+b).$$

**Example 20** Find the derivative of $f(x) = \sqrt{\sin x}$, by first principle.

**Solution** By definition,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{\sin(x+h)} - \sqrt{\sin x}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{\sin(x+h)} - \sqrt{\sin x}\right)\left(\sqrt{\sin(x+h)} + \sqrt{\sin x}\right)}{h\left(\sqrt{\sin(x+h)} + \sqrt{\sin x}\right)}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h\left(\sqrt{\sin(x+h)} + \sqrt{\sin x}\right)}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}}{2 \cdot \frac{h}{2}\left(\sqrt{\sin(x+h)} + \sqrt{\sin x}\right)}$$

$$= \frac{\cos x}{2\sqrt{\sin x}} = \frac{1}{2}\cot x\sqrt{\sin x}$$

**Example 21** Find the derivative of $\frac{\cos x}{1+\sin x}$.

**Solution** Let $y = \frac{\cos x}{1+\sin x}$

Differentiating both sides with respects to $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cos x}{1+\sin x}\right)$$

$$= \frac{(1+\sin x)\frac{d}{dx}(\cos x) - \cos x\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-(1+\sin x)}{(1+\sin x)^2} = \frac{-1}{1+\sin x}$$

**Objective Type Questions**

Choose the correct answer out of the four options given against each Example 22 to 28 (M.C.Q.).

**Example22** $\lim\limits_{x\to 0} \frac{\sin x}{x(1+\cos x)}$ is equal to

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) –1

**Solution** (B) is the correct answer, we have

$$\lim_{x\to 0} \frac{\sin x}{x(1+\cos x)} = \lim_{x\to 0} \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{x\left(2\cos^2\frac{x}{2}\right)}$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{\tan\frac{x}{2}}{\frac{x}{2}} = \frac{1}{2}$$

**Example23** $\lim\limits_{x\to \frac{\pi}{2}} \frac{1-\sin x}{\cos x}$ is equal to

(A) 0 (B) –1 (C) 1 (D) does not exit

**Solution** (A) is the correct answer, since

$$\lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{\cos x} = \lim_{y\to 0}\left[\frac{1-\sin\left(\frac{\pi}{2}-y\right)}{\cos\left(\frac{\pi}{2}-y\right)}\right] \left(\text{taking } \frac{\pi}{2}-x=y\right)$$

$$= \lim_{y\to 0} \frac{1-\cos y}{\sin y} = \lim_{y\to 0} \frac{2\sin^2 \frac{y}{2}}{2\sin\frac{y}{2}\cos\frac{y}{2}}$$

$$= \lim_{y\to 0} \tan\frac{y}{2} = 0$$

**Example 24** $\lim_{x\to 0} \frac{|x|}{x}$ is equal to

(A) 1 (B) –1 (C) 0 (D) does not exists

**Solution** (D) is the correct answer, since

$$\text{R.H.S} = \lim_{x\to 0^+} \frac{|x|}{x} = \frac{x}{x} = 1$$

and

$$\text{L.H.S} = \lim_{x\to 0^-} \frac{|x|}{x} = \frac{-x}{x} = -1$$

**Example 25** $\lim_{x\to 1} [x-1]$, where [.] is greatest integer function, is equal to

(A) 1 (B) 2 (C) 0 (D) does not exists

**Solution** (D) is the correct answer, since

$$\text{R.H.S} = \lim_{x\to 1^+} [x-1] = 0$$

and

$$\text{L.H.S} = \lim_{x\to 1^-} [x-1] = -1$$

**Example 26** $\lim_{x\to 0} x\sin\frac{1}{x}$ is equals to

(A) 0 (B) 1 (C) $\frac{1}{2}$ (D) does not exist

**Solution** (A) is the correct answer, since

$\lim_{x\to 0} x = 0$ and $-1 \leq \sin\frac{1}{x} \leq 1$, by Sandwitch Theorem, we have

$$\lim_{x \to 0} x \sin\frac{1}{x} = 0$$

**Example 27** $\lim_{n \to \infty} \frac{1+2+3+\ldots+n}{n^2}$, $n \in \mathbf{N}$, is equal to

(A) 0 (B) 1 (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

**Solution** (C) is the correct answer. As $\lim_{x \to \infty} \frac{1+2+3+\ldots+n}{n^2}$

$$= \lim_{n \to \infty} \frac{n(n+1)}{2n^2} = \lim_{x \to \infty} \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2}$$

**Example 28** If $f(x) = x \sin x$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

(A) 0 (B) 1 (C) –1 (D) $\frac{1}{2}$

**Solution** (B) is the correct answer. As $f'(x) = x \cos x + \sin x$

So, $$f'\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 1$$

## 13.3 EXERCISE

**Short Answer Type**

Evaluate :

1. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$
2. $\lim_{x \to \frac{1}{2}} \frac{4x^2 - 1}{2x - 1}$
3. $\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$
4. $\lim_{x \to 0} \frac{(x+2)^{\frac{1}{3}} - 2^{\frac{1}{3}}}{x}$
5. $\lim_{x \to 1} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$
6. $\lim_{x \to a} \frac{(2+x)^{\frac{5}{2}} - (a+2)^{\frac{5}{2}}}{x - a}$

**7.** $\lim_{x \to 1} \frac{x^4 - \sqrt{x}}{\sqrt{x} - 1}$

**8.** $\lim_{x \to 2} \frac{x^2 - 4}{\sqrt{3x - 2} - \sqrt{x + 2}}$

**9.** $\lim_{x \to \sqrt{2}} \frac{x^4 - 4}{x^2 + 3\sqrt{2}x - 8}$

**10.** $\lim_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$

**11.** $\lim_{x \to 0} \frac{\sqrt{1 + x^3} - \sqrt{1 - x^3}}{x^2}$

**12.** $\lim_{x \to -3} \frac{x^3 + 27}{x^5 + 243}$

**13.** $\lim_{x \to \frac{1}{2}} \left( \frac{8x - 3}{2x - 1} - \frac{4x^2 + 1}{4x^2 - 1} \right)$

**14.** Find '$n$', if $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80$, $n \in \mathbf{N}$

**15.** $\lim_{x \to a} \frac{\sin 3x}{\sin 7x}$

**16.** $\lim_{x \to 0} \frac{\sin^2 2x}{\sin^2 4x}$

**17.** $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$

**18.** $\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3}$

**19.** $\lim_{x \to 0} \frac{1 - \cos mx}{1 - \cos nx}$

**20.** $\lim_{x \to \frac{\pi}{3}} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2}\left(\frac{\pi}{3} - x\right)}$

**21.** $\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$

**22.** $\lim_{x \to \frac{\pi}{6}} \frac{\sqrt{3} \sin x - \cos x}{x - \frac{\pi}{6}}$

**23.** $\lim_{x \to 0} \frac{\sin 2x + 3x}{2x + \tan 3x}$

**24.** $\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$

**25.** $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

**26.** $\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$

**27.** $\lim_{x \to 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x}$

**28.** If $\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$, then find the value of $k$.

Differentiate each of the functions w. r. to $x$ in Exercises 29 to 42.

**29.** $\frac{x^4 + x^3 + x^2 + 1}{x}$

**30.** $\left( x + \frac{1}{x} \right)^3$

**31.** $(3x + 5)(1 + \tan x)$

**32.** $(\sec x - 1)(\sec x + 1)$ **33.** $\dfrac{3x+4}{5x^2-7x+9}$ **34.** $\dfrac{x^5-\cos x}{\sin x}$

**35.** $\dfrac{x^2\cos\frac{\pi}{4}}{\sin x}$ **36.** $(ax^2+\cot x)(p+q\cos x)$

**37.** $\dfrac{a+b\sin x}{c+d\cos x}$ **38.** $(\sin x+\cos x)^2$ **39.** $(2x-7)^2(3x+5)^3$

**40.** $x^2\sin x+\cos 2x$ **41.** $\sin^3 x\cos^3 x$ **42.** $\dfrac{1}{ax^2+bx+c}$

## Long Answer Type

Differentiate each of the functions with respect to '$x$' in Exercises 43 to 46 using first principle.

**43.** $\cos(x^2+1)$ **44.** $\dfrac{ax+b}{cx+d}$ **45.** $x^{\frac{2}{3}}$

**46.** $x\cos x$

Evaluate each of the following limits in Exercises 47 to 53.

**47.** $\lim\limits_{y\to 0}\dfrac{(x+y)\sec(x+y)-x\sec x}{y}$

**48.** $\lim\limits_{x\to 0}\dfrac{(\sin(\alpha+\beta)x+\sin(\alpha-\beta)x+\sin 2\alpha x)}{\cos 2\beta x-\cos 2\alpha x}\cdot x$

**49.** $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\tan^3 x-\tan x}{\cos\left(x+\frac{\pi}{4}\right)}$ **50.** $\lim\limits_{x\to\pi}\dfrac{1-\sin\frac{x}{2}}{\cos\frac{x}{2}\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)}$

**51.** Show that $\lim\limits_{x\to 4}\dfrac{|x-4|}{x-4}$ does not exists

**52.** Let $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x} & \text{when } x \neq \dfrac{\pi}{2} \\ 3 & x = \dfrac{\pi}{2} \end{cases}$ and if $\lim\limits_{x \to \frac{\pi}{2}} f(x) = f(\frac{\pi}{2})$,

find the value of $k$.

**53.** Let $f(x) = \begin{cases} x + 2 & x \leq -1 \\ cx^2 & x > -1 \end{cases}$, find '$c$' if $\lim\limits_{x \to -1} f(x)$ exists.

### Objective Type Questions

Choose the correct answer out of 4 options given against each Exercise 54 to 76 (M.C.Q).

**54.** $\lim\limits_{x \to \pi} \dfrac{\sin x}{x - \pi}$ is

(A) 1 (B) 2 (C) –1 (D) –2

**55.** $\lim\limits_{x \to 0} \dfrac{x^2 \cos x}{1 - \cos x}$ is

(A) 2 (B) $\dfrac{3}{2}$ (C) $\dfrac{-3}{2}$ (D) 1

**56.** $\lim\limits_{x \to 0} \dfrac{(1+x)^n - 1}{x}$ is

(A) $n$ (B) 1 (C) $-n$ (D) 0

**57.** $\lim\limits_{x \to 1} \dfrac{x^m - 1}{x^n - 1}$ is

(A) 1 (B) $\dfrac{m}{n}$ (C) $-\dfrac{m}{n}$ (D) $\dfrac{m^2}{n^2}$

**58.** $\lim\limits_{x \to 0} \dfrac{1 - \cos 4\theta}{1 - \cos 6\theta}$ is

(A) $\frac{4}{9}$ (B) $\frac{1}{2}$ (C) $\frac{-1}{2}$ (D) $-1$

**59.** $\lim_{x \to 0} \frac{\operatorname{cosec} x - \cot x}{x}$ is

(A) $\frac{-1}{2}$ (B) 1 (C) $\frac{1}{2}$ (D) 1

**60.** $\lim_{x \to 0} \frac{\sin x}{\sqrt{x+1} - \sqrt{1-x}}$ is

(A) 2 (B) 0 (C) 1 (D) $-1$

**61.** $\lim_{x \to \frac{\pi}{4}} \frac{\sec^2 x - 2}{\tan x - 1}$ is

(A) 3 (B) 1 (C) 0 (D) $\sqrt{2}$

**62.** $\lim_{x \to 1} \frac{\left(\sqrt{x} - 1\right)\left(2x - 3\right)}{2x^2 + x - 3}$ is

(A) $\frac{1}{10}$ (B) $\frac{-1}{10}$ (C) 1 (D) None of these

**63.** If $f(x) = \begin{cases} \frac{\sin[x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$, where [.] denotes the greatest integer function ,

then $\lim_{x \to 0} f(x)$ is equal to

(A) 1 (B) 0 (C) $-1$ (D) None of these

**64.** $\lim_{x \to 0} \frac{|\sin x|}{x}$ is

(A) 1 (B) $-1$ (C) does not exist (D) None of these

**65.** Let $f(x) = \begin{cases} x^2 - 1, & 0 < x < 2 \\ 2x + 3, & 2 \leq x < 3 \end{cases}$, the quadratic equation whose roots are $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$ is

(A) $x^2 - 6x + 9 = 0$ (B) $x^2 - 7x + 8 = 0$

(C) $x^2 - 14x + 49 = 0$ (D) $x^2 - 10x + 21 = 0$

**66.** $\lim\limits_{x \to 0} \dfrac{\tan 2x - x}{3x - \sin x}$ is

(A) 2 (B) $\dfrac{1}{2}$ (C) $\dfrac{-1}{2}$ (D) $\dfrac{1}{4}$

**67.** Let $f(x) = x - [x]; \in \mathbf{R}$, then $f'\left(\dfrac{1}{2}\right)$ is

(A) $\dfrac{3}{2}$ (B) 1 (C) 0 (D) –1

**68.** If $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, then $\dfrac{dy}{dx}$ at $x = 1$ is

(A) 1 (B) $\dfrac{1}{2}$ (C) $\dfrac{1}{\sqrt{2}}$ (D) 0

**69.** If $f(x) = \dfrac{x-4}{2\sqrt{x}}$, then $f'(1)$ is

(A) $\dfrac{5}{4}$ (B) $\dfrac{4}{5}$ (C) 1 (D) 0

**70.** If $y = \dfrac{1 + \dfrac{1}{x^2}}{1 - \dfrac{1}{x^2}}$, then $\dfrac{dy}{dx}$ is

(A) $\dfrac{-4x}{(x^2-1)^2}$ (B) $\dfrac{-4x}{x^2-1}$ (C) $\dfrac{1-x^2}{4x}$ (D) $\dfrac{4x}{x^2-1}$

**71.** If $y = \dfrac{\sin x + \cos x}{\sin x - \cos x}$, then $\dfrac{dy}{dx}$ at $x = 0$ is

(A) $-2$ (B) 0 (C) $\frac{1}{2}$ (D) does not exist

**72.** If $y = \frac{\sin(x+9)}{\cos x}$ then $\frac{dy}{dx}$ at $x = 0$ is

(A) $\cos 9$ (B) $\sin 9$ (C) 0 (D) 1

**73.** If $f(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^{100}}{100}$, then $f'(1)$ is equal to

(A) $\frac{1}{100}$ (B) 100 (C) does not exist (D) 0

**74.** If $f(x) = \frac{x^n - a^n}{x - a}$ for some constant '$a$', then $f'(a)$ is

(A) 1 (B) 0 (C) does not exist (D) $\frac{1}{2}$

**75.** If $f(x) = x^{100} + x^{99} + \ldots + x + 1$, then $f'(1)$ is equal to

(A) 5050 (B) 5049 (C) 5051 (D) 50051

**76.** If $f(x) = 1 - x + x^2 - x^3 \ldots - x^{99} + x^{100}$, then $f'(1)$ is euqal to

(A) 150 (B) $-50$ (C) $-150$ (D) 50

**Fill in the blanks in Exercises 77 to 80.**

**77.** If $f(x) = \frac{\tan x}{x - \pi}$, then $\lim_{x \to \pi} f(x) =$ ______________

**78.** $\lim_{x \to 0} \left( \sin mx \cot \frac{x}{\sqrt{3}} \right) = 2$, then $m =$ ______________

**79.** if $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, then $\frac{dy}{dx} =$ ______________

**80.** $\lim_{x \to 3^+} \frac{x}{[x]} =$ ______________